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Delta-convergent sequences that vanish at the support of the limit Dirac delta function

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Abstract
The known limit representations of the Dirac delta function involve delta sequences that do not vanish at the support of the limit Dirac delta function. However, Galapon (2009 *Proc. R. Soc.* A **465** 71) assumes that a delta sequence may vanish at the support of the limit Dirac delta function for all finite values of the limit parameter. Here, such delta sequences are shown to exist by construction.

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In [1] we developed a theory of quantum first time of arrival and considered the appearance of particle in the theory. Numerical simulation yielded a sequence of normalized empirical functions, $\{g_\nu(x)\}$, with $g_\nu(0) = 0$ for all finite ν and with $g_\nu(x)$ having two dominant peaks coalescing at $x = 0$ as ν gets arbitrarily large [1, 2]. (See figure 1(*b*).) We claimed that the sequence tended to the Dirac delta function (DDF) $\delta(x)$. If $g_\nu(x)$ indeed tends to $\delta(x)$ as ν approaches infinity, then $g_\nu(x)$ has the property that it vanishes at the support of its limit Dirac delta function. Since we are not aware of a sequence converging to the Dirac delta function with such a property, it becomes important to demonstrate the existence of such sequences to lend a plausibility proof to our claim in [1, 2]. In this paper, we give examples of sequences of functions that vanish at the support of their limit Dirac delta function.

The DDF $\delta(x)$ is defined by the formal property $\int_{-\infty}^{\infty}\delta(x)\psi(x)\,\mathrm{d}x = \psi(0)$, for sufficiently well-behaved functions $\psi(x)$. Rigorously, the DDF is defined by means of a sequence of integrable functions, $\{f_\nu(x)\}$, with the property that $\lim_{\nu\to\infty}\int_{-\infty}^{\infty} f_\nu(x)\psi(x)\,\mathrm{d}x = \psi(0)$. Such a sequence is referred to as a delta-convergent sequence or simply a delta sequence. For a given delta sequence, $\{f_\nu(x)\}$, we have what is known as a limit representation of the DDF, $\delta(x) = \lim_{\nu\to\infty} f_\nu(x)$. Known delta sequences have either increasing positive or infinite values at the origin; that is, for sufficiently large ν, $0 < f_\nu(0) < f_{\nu'}(0)$ when $\nu < \nu'$ or $f_\nu(0) = \infty$ for all ν. For example, for the former, we have the well-known representations $\lim_{\nu\to\infty}\sin(\nu x)(\pi x)^{-1} = \delta(x)$ and $\lim_{\nu\to\infty}\nu[\pi(1+\nu^2x^2)]^{-1} = \delta(x)$ [3]; for the latter, we

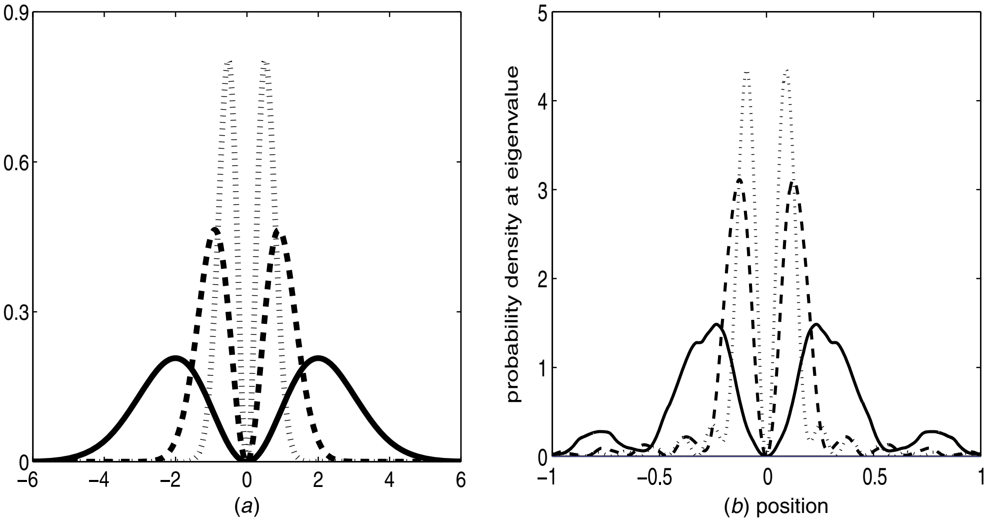

Figure 1. (*a*) Plot of $h_\nu(1, x)$ for $\nu = 1$ (solid line), $\nu = 5$ (dashed line) and $\nu = 15$ (dotted line). (*b*) The empirical sequence claimed in [1] that tends to the Dirac delta function.

have $\lim_{\nu\to\infty} |x|^{-1+1/\nu}(2\nu)^{-1} = \delta(x)$ [4]. We are not aware of delta sequences with the property that $f_\nu(0) = 0$ for all finite ν.

To prove the existence of delta sequences that vanish at the support of the limit Dirac delta function, we will need the following well-known theorem.

Theorem 1 ([5]). *A sequence of functions,* $\{f_\nu(x)\}$*, is a delta-convergent sequence if (1) for any* $M > 0$ *and for* $|a|, |b| \leqslant M$*, the quantities* $\left|\int_a^b f_\nu(x)\,\mathrm{d}x\right|$ *are bounded by a constant depending only on M, and (2) for any fixed non-vanishing a and b*

$$\lim_{\nu\to\infty}\int_a^b f_\nu(x)\,\mathrm{d}x = \begin{cases} 0, & \text{for } \quad 0 < a < b \qquad \text{and} \quad a < b < 0 \\ 1, & \text{for } \quad a < 0 < b. \end{cases}$$

Using this theorem, we now wish to show that for any positive integer n the set of functions

$$h_\nu(n, x) = \frac{1}{2^{2n+1}\Gamma(n + 1/2)}\nu^{n+\frac{1}{2}} x^{2n}\,\mathrm{e}^{-x^2\nu/4}$$

is a delta sequence in ν so that $\lim_{\nu\to\infty} h_\nu(n, x) = \delta(x)$. These functions satisfy $h_\nu(n, 0) = 0$ for all $0 < \nu < \infty$, and have two peaks that coalesce at $x = 0$ as ν tends to infinity. (See figure 1(*a*).)

First, for a fixed positive integer n the sequence $\left|\int_a^b h_\nu(n, x)\,\mathrm{d}x\right|$ is uniformly bounded for any a and b. This is the case because $h_\nu(n, x)$'s are normalized, which can be shown by changing variables from x to $y = x\sqrt{\nu}/2$:

$$\int_{-\infty}^{\infty} h_\nu(n, x)\,\mathrm{d}x = \frac{1}{\Gamma(n + 1/2)}\int_{-\infty}^{\infty} y^{2n}\,\mathrm{e}^{-y^2}\mathrm{d}y = 1,$$

where we have used the identity $\int_{-\infty}^{\infty} y^{2n}\,\mathrm{e}^{-y^2}\,\mathrm{d}y = \Gamma(n + 1/2)$ for $n = 0, 1, 2, \ldots$ [6]. Then, for any a and b, we have $\left|\int_a^b h_\nu(n, x)\,\mathrm{d}x\right| \leqslant 1$ for all ν, so that $\left|\int_a^b h_\nu(n, x)\,\mathrm{d}x\right|$ is bounded by a constant independent of a, b and ν. Now for $0 < a < b$, we have the integral

$$\int_a^b h_\nu(n, x)\,\mathrm{d}x = \frac{1}{\Gamma(n + 1/2)}\int_{a\sqrt{\nu}/2}^{b\sqrt{\nu}/2} y^{2n}\,\mathrm{e}^{-y^2}\,\mathrm{d}y \leqslant \int_{a\sqrt{\nu}/2}^{\infty} y^{2n}\,\mathrm{e}^{-y^2}\,\mathrm{d}y,$$

where we have effected the same change in variables. The right-hand side of the inequality tends to zero as ν tends to infinity; hence, $\lim_{\nu\to\infty}\int_a^b h_\nu(n,x) = 0$. Similar treatment for $b < a < 0$ gives the same conclusion that the integral vanishes in the limit. On the other hand, for $a < 0 < b$, we have

$$\begin{aligned}\int_a^b h_\nu(n,x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} h_\nu(n,x)\,\mathrm{d}x - \int_{-\infty}^{a} h_\nu(n,x)\,\mathrm{d}x - \int_{b}^{\infty} h_\nu(n,x)\,\mathrm{d}x \\ &= 1 - \frac{1}{\Gamma(n+1/2)}\left[\int_{|a|\sqrt{\nu}/2}^{\infty} y^{2n}\,\mathrm{e}^{-y^2}\,\mathrm{d}y + \int_{b\sqrt{\nu}/2}^{\infty} y^{2n}\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right].\end{aligned}$$

The bracketed integrals tend to zero as ν tends to infinity; hence, $\lim_{\nu\to\infty}\int_a^b h_\nu(n,x) = 1$. Thus for every positive integer n, the sequence $\{h_\nu(n,x)\}$ is a delta sequence. Other limit representations can be constructed from $h_\nu(n,x)$'s.

We now describe how. Let $f_\nu(1,x), f_\nu(2,x), f_\nu(3,x), \ldots$ be a sequence of delta sequences in ν, where $\int_{-\infty}^{\infty} f_\nu(n,x)\,\mathrm{d}x = 1$ for every positive integer n, e.g. $h_\nu(n,x)$'s. Let $\{p_n\}$ be a sequence of real numbers such that the sum $\sum_{n=1}^{\infty} p_n$ is absolutely convergent. Moreover, let $\sum_{n=1}^{\infty} p_n f_\nu(n,x)$ be uniformly convergent in x. Then the function

$$f_\nu(x) = \frac{\sum_{n=1}^{\infty} p_n f_\nu(n,x)}{\sum_{n=1}^{\infty} p_n}$$

is a delta sequence in ν. This follows because the uniform convergence of the sum allows term-by-term integration of the series; moreover, the uniform boundedness of the integral $a_n(\nu) = \int_a^b f_\nu(n,x)\,\mathrm{d}x$, $|a_n(\nu)| \leqslant 1$, yields the uniformly convergent sum $\sum_{n=1}^{\infty} p_n a_n(\nu)$ in ν, from which we can interchange the order of the sum and the limit for ν. That $f_\nu(x)$ is a delta sequence now follows immediately from the fact that it is normalized by construction. Now more representations can be obtained by taking subsequences of the sequence $f_\nu(n,x)$ and then treating these subsequences as another sequence of delta sequences; we can then proceed as described.

This procedure allows us to construct a host of limit representations of the Dirac delta function from $h_\nu(n,x)$'s. Given the sequence $\{p_n\}$, with $\sum_{k=1}^{\infty} p_n$ absolutely convergent, the series

$$h_\nu(x) = \frac{\sum_{n=1}^{\infty} p_n h_\nu(n,x)}{\sum_{n=1}^{\infty} p_n} = \frac{1}{\sum_{n=1}^{\infty} p_n}\sum_{n=1}^{\infty} p_n \frac{1}{2^{2n+1}\Gamma(n+1/2)}\nu^{n+\frac{1}{2}} x^{2n}\,\mathrm{e}^{-x^2\nu/4}$$

is uniformly convergent. To prove its uniform convergence, note that for every x we have the bound $\nu^n x^{2n}\,\mathrm{e}^{-x^2\nu/4} \leqslant 4^n n^n\,\mathrm{e}^{-n}$; then

$$|p_n h_\nu(n,x)| \leqslant \frac{|p_n|}{2\Gamma(n+1/2)}\nu^{\frac{1}{2}} n^n\,\mathrm{e}^{-n}.$$

By the Weierstrass M-test, the series $h_\nu(x)$ converges uniformly if there exists a sequence of positive numbers $\{M_n\}$ such that $|p_n h_\nu(n,x)| \leqslant M_n$ for all all x and n, with $\sum_{n=1}^{\infty} M_n$ absolutely convergent. From the above inequality bounding $|p_n h_\nu(n,x)|$, we find that such a sequence is given by

$$M_n = \frac{|p_n|}{2\Gamma(n+1/2)}\nu^{\frac{1}{2}} n^n\,\mathrm{e}^{-n}, \qquad n = 1, 2, \ldots.$$

The convergence of the sum of M_n's follows from the absolute convergence of the sum of p_n's and the fact that $M_n \sim \sqrt{\nu}|p_n|/2\sqrt{2\pi}$ as n approaches infinity. Hence, the series $h_\nu(x)$ is uniformly convergent and is a delta sequence.

Different limit representations of the DDF with the same properties as those of $h_\nu(n, x)$'s can now be constructed at will. For example, with $p_n = (-1)^n/n!$ and $p_n = 1/2^n$, we have the representations

$$\lim_{\nu\to\infty} \frac{1}{\sqrt{\pi}(1-1/e)} \sqrt{\nu} \sin^2\left(\frac{\sqrt{\nu}x}{2}\right) \mathrm{e}^{-\nu x^2/4} = \delta(x),$$

$$\lim_{\nu\to\infty} \frac{\nu}{4\sqrt{2}} x \,\mathrm{erf}\left(\frac{x}{2}\sqrt{\frac{\nu}{2}}\right) \mathrm{e}^{-\nu x^2/8} = \delta(x),$$

respectively. These delta sequences vanish at the support of the limit Dirac delta function and have two peaks coalescing at $x = 0$ as $\nu \to \infty$. Many more such limit representations of the Dirac delta function can be constructed, but our examples here are sufficient to prove that such delta sequences exist, thus establishing the plausibility of our claim in [1, 2].

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